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A VARIATIONAL THEORY FOR FINITE-STEP ELASTO-PLASTIC PROBLEMS

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Abstract—An extended version of *generalized standard* elasto-plastic material is considered in the framework of an internal variable theory of associated plasticity. According to a backward difference scheme for time integration of the flow rule, a finite-step structural problem is formulated in a geometrically linear range. Convex analysis and a brand new potential theory for monotone multi-valued operators are shown to provide the natural mathematical setting for the derivation of the related variational formulation. A general stationarity principle is obtained and then specialized to obtain a minimum principle in terms of displacements, plastic strains and internal variables. A critical comparison with an analogous minimum principle recently proposed in literature is performed, showing the inadequacy of classical procedures in deriving non-smooth variational formulations.

1. INTRODUCTION

In the early seventies a model of *generalized standard* elasto-plastic material was proposed by Halphen and Nguyen (1975). In this model the plastic flow rule is assigned through a normality law to a generalized elastic domain defined in the product space of stresses and thermodynamic forces. The free energy is assumed to be additively decomposed into two parts depending separately upon elastic strains and internal variables.

An extended version of generalized standard material is here addressed in the framework of convex analysis.

Performing the time integration of the plastic flow rule according to a backward difference scheme, the relevant finite-step elasto-plastic structural problem is formulated in a geometrically linear range.

It is then shown that the associated variational formulation can be developed following the general guidelines provided by recent results presented in Romano *et al.* (1992b, 1993a).

The first step is to recast the structural problem in terms of a multi-valued *structural operator* defined in the product space of all state variables. This operator encompasses in a unique expression the field equations and the constitutive relations describing the finite-step elasto-plastic problem.

The concept of conservativity for multi-valued operators, contributed in Romano *et al.* (1994) is then applied to obtain the related non-smooth potential by direct integration along a ray in the operator domain.

Evaluating the generalized gradient of the non-smooth potential and imposing its stationarity, the operator formulation of the problem is recovered.

Stationarity amounts to requiring that the null vector belongs to the partial subdifferential (superdifferential) of the potential with respect to the arguments in which it results in being convex (concave).

A family of variational principles for the finite-step structural problem can then be obtained by enforcing the fulfilment of field equations and constraint conditions. In particular the explicit derivation of a minimum principle in displacements, plastic strains and internal variables is provided.

Further, by appealing to duality theory in convex programming, a minimum principle in displacements, plastic strains, internal variables and plastic parameters is derived under the assumption of sublinear yield modes.

An analogous minimum principle has been recently proposed in Comi and Maier (1992) and Comi *et al.* (1992), and referred to as non-convex.

A detailed comparison reveals an essential difference in the constraint conditions. In fact the conditions considered in Comi and Maier (1992) and Comi et al. (1992) define a feasible set which turns out to be a proper subset of the variationally consistent one which the development carried out in the present paper leads to. As a consequence a coarser result is obtained and unduly complicated computational algorithms are generated.

This variational flow is shown to be imputable to the mathematical theory resorted to. Classical variational theory, requiring differentiability of the involved functionals, cannot be applied to the treatment of essentially non-smooth problems. A suitable approach must be exploited following the general guidelines provided by concepts and methods of convex analysis and potential theory for monotone multi-valued operators.

2. SOME PRELIMINARY RESULTS

We briefly recall here some basic definitions and properties of convex analysis which will be referred to in the sequel. Further the notion of generalized gradient of a non-smooth potential is introduced.

2.1. *A background of convex analysis*

A comprehensive treatment of the subject can be found in Ekeland and Temam (1976), Ioffe and Tihomirov (1979), Moreau (1966) and Rockafellar (1970).

Let (X, X') be a pair of locally convex topological vector spaces (l.c.t.v.s.) placed in separating duality by a bilinear form $\langle\cdot,\cdot\rangle$.

The one-sided Gateaux derivative of a convex functional $f: X \mapsto \Re \cup \{+\infty\}$ at the point $x_0 \in \operatorname{dom} f$, along the direction defined by the vector $x \in X$, is given by:

$$\mathrm{d}f(x_0; x) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}[f(x_0+\varepsilon x) - f(x_0)].$$

The limit exists at every point $x_0 \in \operatorname{dom} f$ along any direction $x \in X$ since the difference quotient in the definition above does not increase as ε decreases to zero [see Ioffe and Tihomirov (1979) and Rockafellar (1970)].

Let us set $\bar{\Re} = \{-\infty\} \cup \Re \cup \{+\infty\}$. The functional $s: X \mapsto \bar{\Re}$ defined by:

$$s(x) \stackrel{\text{def}}{=} \mathrm{d}f(x_0; x),$$

turns out to be *sublinear*, that is positively homogeneous and subadditive:

$$\begin{cases} s(\alpha x) = \alpha s(x) & \forall \alpha \geqslant 0 \quad \text{(positive homogeneity)} \\ s(x_1) + s(x_2) \geqslant s(x_1 + x_2) & \forall x_1, x_2 \in X \quad \text{(subadditivity)} \end{cases}$$

Clearly the epigraph of s is a convex cone in $X \times \Re$.

If the sublinear functional s is proper, that is nowhere $-\infty$, and lower semicontinuous (l.s.c.):

$$\liminf_{z \to x} s(z) = s(x) \quad \forall z \in X,$$

it turns out to be the support functional of a non-empty closed convex set K:

$$s(x) = \sup\{\langle x^*, x\rangle \mid x^* \in K\},$$

with

$$K = \{x^* \in X' \mid s(x) \geqslant \langle x^*, x\rangle \quad \forall x \in X\}.$$

A proper convex functional is closed if and only if it is l.s.c.

The *subdifferential* of the functional f is the multi-valued map, $\partial f: X \mapsto X'$, defined by:

$$\partial f(x_0) \stackrel{\text{def}}{=} K,$$

and the elements of K are called subgradients.

In particular, if the functional f is differentiable at $x_0 \in X$, the subdifferential is a singleton and coincides with the usual differential.

We remark that the above definition of subdifferential turns out to be equivalent to the usual definition of subdifferential in convex analysis (Rockafellar, 1970), that is:

$$x^* \in \partial f(x_0) \Leftrightarrow f(y) - f(x_0) \geqslant \langle x^*, y - x_0 \rangle \quad \forall\, y \in X.$$

The following rules usually hold for subdifferentiability (Ioffe and Tihomirov, 1979; Romano *et al.*, 1992c):

(1) *Chain-rule:* given a differentiable operator $A: X \mapsto Y$ and a convex functional $f: Y \mapsto \Re \cup \{+\infty\}$ which is subdifferentiable at $y = A(x)$ we have:

$$\partial(f \circ A)(x) = [\mathrm{d}A(x)]' \, \partial f(A(x)),$$

where $\mathrm{d}A(x)$ is the derivative of the operator A at x and $[\mathrm{d}A(x)]'$ is the dual operator;

(2) *Additivity:* given two convex functionals $f_1: X \mapsto \Re \cup \{+\infty\}$ and $f_2: X \mapsto \Re \cup \{+\infty\}$ which are subdifferentiable at $x \in X$, it turns out to be:

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

The conjugate of a convex functional f is the convex functional $f^*: X' \mapsto \Re \cup \{+\infty\}$ defined by:

$$f^*(x^*) = \sup_{y \in X} \{\langle x^*, y \rangle - f(y)\},$$

so that Fenchel's inequality holds:

$$f(y) + f^*(x^*) \geqslant \langle x^*, y \rangle \quad \forall\, y \in X, \quad \forall\, x^* \in X'.$$

The elements x, x^* for which Fenchel's inequality holds as an equality are said to be conjugate and the following relations are equivalent when f is closed:

$$f(x) + f^*(x^*) = \langle x^*, x \rangle, \quad x^* \in \partial f(x), \quad x \in \partial f^*(x^*).$$

Analogous results hold for concave functionals by interchanging the role of $+\infty$, $\geqslant$ and "sup" with those of $-\infty$, $\leqslant$ and "inf"; the prefix "sub" used in the convex case has now to be replaced by "super".

In what follows the subdifferential (superdifferential) of a convex (concave) functional will be denoted by the same symbol ∂ when no ambiguity can arise.

A relevant example of conjugate functionals associated with a convex set K is provided by the *indicator* functional:

$$\sqcup_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise,} \end{cases}$$

and by the *support* functional:

$$\sqcup_K^*(x^*) = \sup_{x \in K} \langle x^*, x \rangle.$$

Moreover we recall that the subdifferential of the indicator functional of a convex set K at a point $x \in K$ coincides with the *normal cone* to K at x:

$$\partial \sqcup_K(x) = N_K(x) \overset{\text{def}}{=} \begin{cases} \{x^* \in X' : \langle x^*, y-x \rangle \leqslant 0 \quad \forall y \in X\}, & \text{if } x \in K \\ \varnothing & \text{otherwise.} \end{cases}$$

2.2. *Potential theory, generalized gradients and stationarity properties*

In the classical calculus of variations the stationarity condition for a differentiable functional amounts to finding its *critical points*, that is the points in the domain of the functional at which its variations, i.e. two-sided directional derivatives, vanish.

Such concepts must be suitably generalized when multi-valued operators and non-smooth functionals are dealt with.

A potential theory for such operators has been recently contributed in Romano *et al.* (1994) which the interested reader is referred to for a detailed account. We shall here summarize the main results which will be of use in the sequel.

A basic result of this theory ensures that the integral of monotone multi-valued operators along any straight line can be unambiguously defined.

The classical concept of conservativity can then be suitably extended by requiring that the integral of a monotone multi-valued operator vanishes along any closed polyline in its domain. The related potential can thus be evaluated by integrating along a straight segment in its domain.

The integral theorem provided in Romano *et al.* (1994) ensures further that a multi-valued operator which is directly expressed as subdifferential of a convex functional, turns out to be monotone and conservative; its potential coincides with the functional itself, to within an additive constant.

Hence, given a convex functional $f: X \mapsto \Re \cup \{+\infty\}$ and denoting by $G = \partial f: X \mapsto X'$, its subdifferential turns out to be:

$$g(x) - g(x_0) = \int_{x_0}^{x} \langle G(\xi), \mathrm{d}\xi \rangle = \int_0^1 \langle G(x_0 + t(x-x_0)), (x-x_0) \rangle \, \mathrm{d}t,$$

and

$$g(x) = f(x) + \text{constant}.$$

We shall say that a convex functional f has a *stationary point* at $x \in \operatorname{dom} f$ if the null vector $0 \in X'$ belongs to its subdifferential at x:

$$0 \in \partial f(x) \Leftrightarrow \mathrm{d}f(x; h) \geqslant 0 \quad \forall h \in X.$$

The concepts above can be further generalized to the case of functionals which are convex with respect to some variables and concave with respect to others.

To fix the ideas let us consider a functional $f: X \times Y \mapsto \Re$ which can be written as the sum of a convex functional $f_1: X \mapsto \Re \cup \{+\infty\}$ and of a concave one $f_2: Y \mapsto \Re \cup \{-\infty\}$, i.e.

$$f(x, y) \overset{\text{def}}{=} f_1(x) + f_2(y).$$

Setting $G_1(x) = \partial f_1(x)$ and $G_2(y) = \partial f_2(y)$ we then consider the product operator:

$$G(x, y) \overset{\text{def}}{=} G_1(x) \times G_2(y) \subseteq X' \times Y'.$$

It is easy to see that the multi-valued operator G is conservative and that its potential is given by the sum of the potential of G_1 and G_2. In fact, taking account of the definition of the duality pairing in the product space $X \times Y$, we can integrate along a straight segment in $X \times Y$ to get:

$$\int_{(x_0,y_0)}^{(x,y)} \langle G(\xi,\eta), \mathrm{d}(\xi,\eta)\rangle = \int_{(x_0,y_0)}^{(x,y)} \langle G_1(\xi)\times G_2(\eta), \mathrm{d}\xi \times \mathrm{d}\eta\rangle$$
$$= \int_{x_0}^{x} \langle G_1(\xi), \mathrm{d}\xi\rangle + \int_{y_0}^{y} \langle G_2(\eta), \mathrm{d}\eta\rangle.$$

The integral of G along any closed polyline then vanishes by virtue of the conservativity of G_1 and G_2.

From the formula above we also infer that the potential of G is given by:

$$\int_{(x_0,y_0)}^{(x,y)} \langle G(\xi,\eta), \mathrm{d}(\xi,\eta)\rangle = f_1(x) - f_1(x_0) + f_2(y) - f_2(y_0)$$
$$= f(x,y) - f(x_0,y_0).$$

The multi-valued operator G is then consistently termed *generalized gradient* of the saddle functional f and we shall write:

$$G = \partial f.$$

The stationarity of f is thus enforced by the condition:

$$(0,0) \in \partial f(x,y) \Leftrightarrow \begin{cases} 0 \in \partial f_1(x), \\ 0 \in \partial f_2(y), \end{cases}$$

where the upper ∂ denotes the subdifferential and the lower one the superdifferential.

A further item in the subdifferential theory of convex functionals has to be stressed. If $f: X \mapsto \Re \cup \{+\infty\}$ is a convex functional and the ambient space turns out to be the Cartesian product of n component spaces, i.e. $X = X_1 \times \cdots \times X_n$, the global subdifferential of f with respect to $x \in X$ is not equal, in general, to the Cartesian product of the partial subdifferentials with respect to each argument $x_i \in X_i$.

In fact the following general inclusion holds (Romano *et al.*, 1993a):

$$\partial f(x) \subseteq \prod_{i=1}^{n} \partial_{x_i} f(x) \overset{\mathrm{def}}{=} \partial_{x_1} f(x) \times \cdots \times \partial_{x_n} f(x), \quad x = (x_1, \ldots, x_n).$$

However, equality holds in the special case in which f is the sum of n convex functionals f_i and each f_i depends only on the corresponding argument x_i:

$$f(x) = \sum_{i=1}^{n} f_i(x_i).$$

Then we have:

$$\partial f(x) = \prod_{i=1}^{n} \partial f_i(x_i),$$

so that the stationarity (minimum) condition on f can accordingly be enforced on each f_i separately:

$$0 \in \partial f(x) \Leftrightarrow 0 \in \partial f_i(x_i), \quad i = 1, \ldots, n.$$

Clearly analogous results can be stated for concave functionals.

3. CONSTITUTIVE RELATIONS

Let us consider a continuous elasto-plastic body undergoing small deformations in an isothermal process.

We assume that hardening phenomena are described by means of a set of internal parameters which take account of the significant changes in the material structure at the microscale level. Contributions to this issue which are more relevant to the present treatment have been given by Halphen and Nguyen (1975), Nguyen (1977), Martin (1975, 1981), Martin and Reddy (1988), Martin and Nappi (1990), Eve *et al.* (1990) and Romano *et al.* (1992a, 1993b).

We address a constitutive model which is an extension of the *generalized standard* elasto-plastic material proposed by Halphen and Nguyen (1975).

Accordingly, the existence of a generalized elastic domain C_l in the product space of stresses and thermodynamic forces $\mathcal{S}_l \times X'_l$ is postulated. This set is assumed to be convex and to contain the origin.

The subscript "l" is used to distinguish the local entities, such as the variables appearing in the constitutive relations, from the corresponding global fields pertaining to the whole structure. In this respect we shall denote by x the points in the domain V of the body.

Denoting by $\mathcal{D}_l$ the linear space of strains $\varepsilon(x)$ and by X_l the linear space of internal variables $\alpha(x)$, the corresponding duals will be the linear space $\mathcal{S}_l$ of internal stresses $\sigma(x)$ and the linear space X'_l of thermodynamic forces $\chi(x)$. As usual the total strain $\varepsilon(x) \in \mathcal{D}_l$ is assumed to be the sum of an elastic strain $e(x)$ and of a plastic strain $p(x)$:

$$\varepsilon(x) = e(x) + p(x).$$

The flow rule assumed by Halphen and Nguyen (1975) for the *generalized standard* elasto-plastic material is the counterpart of the analogous expression for the classical standard material in which the elastic domain is considered in the space of stresses alone.

Actually the flow rule is expressed by the condition that the right time derivatives of the plastic strains and of the internal variables $\dot{p}(x), \dot{\alpha}(x)$ belong to the normal cone to the local elastic domain C_l at $(\sigma(x), \chi(x))$:

$$[\dot{p}(x), \dot{\alpha}(x)] \in N_{C_l}[\sigma(x), \chi(x)].$$

The free energy function $\varphi : \mathcal{D}_l \times X_l \mapsto \mathbb{R} \cup \{+\infty\}$ is assumed to be jointly convex in the elastic strains $e(x)$ and in the internal variables $\alpha(x)$. Stresses $\sigma(x)$ and thermodynamic forces $\chi(x)$ are accordingly defined by the multi-valued relation:

$$[\sigma(x), -\chi(x)] \in \partial\varphi[e(x), \alpha(x)],$$

where the symbol ∂ denotes the subdifferential operator in the product space of elastic strains and internal variables.

Following Halphen and Nguyen (1975) we shall assume that the free energy is additively decomposed in two parts depending separately upon elastic strains $e(x)$ and internal variables $\alpha(x)$:

$$\varphi[e(x), \alpha(x)] = \phi[e(x)] - \pi[a(x)].$$

The convex function $\phi : \mathcal{D}_l \mapsto \Re \cup \{+\infty\}$ represents the elastic strain energy and the concave function $\pi : X_l \mapsto \Re \cup \{-\infty\}$ describes the role of the internal variables in the hardening processes.

The expression which relates stresses and thermodynamic forces to elastic strains and internal variables can thus be re-written as follows:

$$\begin{cases}\sigma(x) \in \partial\phi[e(x)],\\ \chi(x) \in \partial\pi[\alpha(x)],\end{cases}$$

where the same symbol ∂ denotes the subdifferential operator in the former relation and the superdifferential in the latter one.

4. THE ELASTO-PLASTIC STRUCTURAL PROBLEM

In order to develop the variational formulation of the structural problem the relevant relations must be written in global form, that is in terms of quantities pertaining to the whole structure. In the sequel such quantities will be referred to as *fields*.

For a continuous model such fields are functionals defined in the domain V occupied by the body and are elements of a suitable functional space.

Global functionals are then obtained from the corresponding local functionals through integration over the domain V.

For instance, denoting by Φ the elastic energy stored in the whole structure and by e the elastic strain field, it results:

$$\Phi(e) \stackrel{\text{def}}{=} \int_V \phi[e(x)]\,\mathrm{d}x.$$

Notice that, whenever the local functionals are convex, the corresponding global ones also turn out to be convex in the relevant fields.

The subdifferential of the global elastic energy is defined as:

$$\sigma \in \partial\Phi(e) \stackrel{\text{def}}{\Leftrightarrow} \mathrm{d}\Phi(e\,;\eta) = \int_V \mathrm{d}\phi[e(x)\,;\eta(x)]\,\mathrm{d}x \geqslant \int_V \sigma(x)\cdot[\eta(x)-e(x)]\,\mathrm{d}x \quad \forall\,\eta(x)\in\mathscr{D}_l,$$

where the dot denotes the scalar product between local quantities. Hence the following equivalence holds (Panagiotopoulos, 1985):

$$\sigma \in \partial\Phi(e) \Leftrightarrow \sigma(x) \in \partial\phi[e(x)] \quad \text{almost everywhere in } V.$$

In the sequel we shall denote by Π the global functional corresponding to π and by C the elastic domain of stress and thermodynamic force fields defined by:

$$C = \{(\sigma,\chi)\in\mathscr{S}\times X' \mid [\sigma(x),\chi(x)]\in C_l \quad \forall\,x\in V\}.$$

The description of the elasto-plastic structural model is completed by specifying the field equations and the external constraints.

We make reference to structural models in which equilibrium is unaffected by geometry changes so that a linear strain measure can be adopted.

Let us denote by $\mathscr{U}$, $\mathscr{D}$ and X the linear spaces of displacement, strain and internal variable fields and by $\mathscr{F}$, $\mathscr{S}$ and X' the corresponding duals, that is the linear spaces of external force, stress and thermodynamic force fields, respectively.

In a geometrically linear range, equilibrium and compatibility are expressed as follows:

$$f = T'\sigma \quad \sigma\in\mathscr{S},\ \mathrm{f}\in\mathscr{F},$$

$$\varepsilon = Tu \quad u\in\mathscr{U},\ \varepsilon\in\mathscr{D},$$

where $T:\mathscr{U}\mapsto\mathscr{D}$ and $T':\mathscr{S}\mapsto\mathscr{F}$ are dual linear operators (Panagiotopoulos, 1985; Romano and Rosati, 1988).

External force fields are given by the sum of the applied load l and of reaction fields r of external constraints:

$$f = l+r.$$

Introducing the concave potential $\Upsilon:\mathscr{U}\mapsto\Re\cup\{-\infty\}$, the external constitutive relation between reaction and displacement fields can be written as:

$$r\in\partial\Upsilon(u).$$

Such a relation provides a model of external constraints which includes several cases of interest in Structural Mechanics such as bilateral or unilateral constraints, elastic or elasto-viscoplastic foundations and so on. A survey of the particular expressions assumed by the functional Υ in each of these cases can be found in Rosati (1989).

The expression which relates external force fields to displacement fields is then:

$$f = l+r\in\partial\Gamma(u), \quad \text{or equivalently} \quad u\in\partial\Gamma^*(f),$$

where

$$\Gamma(u) = \langle l,u\rangle+\Upsilon(u), \quad \Gamma^*(f) = \Upsilon^*(f-l),$$

are concave functionals Γ^* and Υ^* being the conjugates of Γ and Υ, respectively.

For the sake of clarity we specialize the general expression of Υ to the particular case of external frictionless bilateral constraints with imposed displacement fields $\bar{u}$.

Denoting by L_0 the subspace of admissible displacement fields and by $R = L_0^\perp$ the subspace of external reaction fields, it results:

$$\Upsilon(u) = \sqcap_{L_0}(u-\bar{u}).$$

Here the symbol $L_0^\perp$ represents the orthogonal complement of the subspace L_0 and $\sqcap$ the concave indicator.

Accordingly the relation $r\in\partial\Upsilon(u)$ is equivalent to stating that $u\in\bar{u}+L_0$ and that $r\in L_0^\perp = R$.

Given a load history $l(t)$, the elasto-plastic evolutive structural problem is thus governed by the following set of relations:

$$\begin{cases} f = T'\sigma & \text{static equilibrium,} \\ e+p = Tu & \text{kinematic compatibility,} \\ (\dot{p},\dot{\alpha})\in N_C(\sigma,\chi) & \text{flow rule,} \\ \sigma\in\partial\Phi(e) & \\ \alpha\in\partial\Pi^*(\chi) & \text{free energy,} \\ u\in\partial\Gamma^*(f) & \text{external constraint,} \end{cases}$$

where Π^* is the conjugate of Π. The explicit dependence of the state variables on time t has been dropped to simplify the notations.

4.1. *Elasto-plastic finite-step structural problem*

The incremental analysis of the non-linear elasto-plastic structural problem is performed by a subdivision of the time interval into a finite number of steps Δt; we assume that no plastic unloading can occur during any of the intervals $\Delta t_i = t_i - t_{i-1}$.

A finite-step analysis of the evolutive problem amounts to evaluating the finite increments of the unknown variables corresponding to a given increment of load when their values are assigned at the beginning of the step. In the sequel we shall denote by $(\cdot)_0$ the known quantities $(\cdot)$ at the beginning of each step.

The irreversible, path-dependent behaviour of plasticity is accounted for by updating the values of the internal variables α at each step.

In order to formulate the finite-step counterpart of the flow rule $(\dot{p}, \dot{\alpha}) \in N_C(\sigma, \chi)$, the time derivative is replaced by the finite increment ratio $(p-p_0, \alpha-\alpha_0)/\Delta t$. Adopting a fully implicit time integration scheme (Euler backward difference), the flow rule is enforced at the end of the step, according to the relation:

$$\frac{1}{\Delta t}(p-p_0, \alpha-\alpha_0) \in N_C(\sigma, \chi),$$

which, being $N_C(\sigma, \chi)$ a convex cone, can also be written:

$$(p-p_0, \alpha-\alpha_0) \in N_C(\sigma, \chi).$$

The extremum characterization of the solution of the finite-step evolutive problem is carried out in the next section.

5. VARIATIONAL FORMULATION

We shall prove in this section that the finite-step elasto-plastic structural problem does admit a variational formulation. In this respect some preliminary considerations can result in a deeper understanding of the matter.

We are interested to provide the expression of a real-valued functional whose gradient, in a suitably generalized sense, yields back the field equations and the constitutive relations governing the structural problem.

As is well known from classical potential theory, an operator admits a potential if and only if it is conservative, i.e. circuital integrals vanish along every closed curve in the connected domain of the operator. The expression of the potential can then be obtained by direct integration. A necessary and sufficient condition for the conservativity of a differentiable operator is the symmetry of its first derivative.

A completely different situation has to be faced when multi-valued operators are dealt with. Elastic–plastic structural problems are relevant examples of this kind since the constitutive relations are expressed as subdifferentials of convex functionals so that dual pairs of state variables are related by monotone multi-valued operators.

A potential theory for such operators has been contributed in Romano *et al.* (1994) and briefly outlined in Section 2.2.

The application of the theory to the finite-step elasto-plastic structural problem will be now exploited in detail. To this end it is convenient to recast the original problem in a suitable operator form.

We recall in advance that the finite-step flow rule can be given the following equivalent formulations:

$$(p-p_0, \alpha-\alpha_0) \in N_C(\sigma, \chi) = \partial \sqcup_C(\sigma, \chi) \Leftrightarrow (\sigma, \chi) \in \partial \sqcup_C^*(p-p_0, \alpha-\alpha_0),$$

where $\sqcup_C^*$ is the support functional of C, defined by:

$$\sqcup_C^*(p-p_0, \alpha-\alpha_0) \stackrel{\text{def}}{=} \sup\{\langle \sigma, p-p_0 \rangle + \langle \chi, \alpha-\alpha_0 \rangle \mid (\sigma, \chi) \in C\}.$$

This functional has the physical meaning of finite-step dissipation associated with a given increment of plastic strain and internal variable fields. It turns out to be non-negative if and only if the null stress and the thermodynamic force fields belong to the elastic domain and strictly positive if they lie in its interior (Romano *et al.*, 1992a).

The dissipation functional pertaining to the whole structure can be evaluated from the local one. In fact, it can be proved that (Panagiotopoulos, 1985; Romano *et al.*, 1992d), from the expression of the local dissipation functional:

$$\sqcup\!\!\lrcorner^*_{C_l}(p(x),\alpha(x)) = \sup\{\sigma(x)\cdot p(x)+\chi(x)\cdot\alpha(x) \mid [\sigma(x),\chi(x)]\in C_l\},$$

the corresponding global one can be obtained by the formula:

$$\sqcup\!\!\lrcorner^*_{C}(p,\alpha) = \int_V \sqcup\!\!\lrcorner^*_{C_l}(p(x),\alpha(x))\,\mathrm{d}x.$$

Adopting the inverse form of the finite-step flow rule involving the global dissipation functional, the finite-step elasto-plastic structural problem can then be re-formulated as follows:

$$\begin{cases} f = T'\sigma, \\ e+p = Tu, \\ (\sigma,\chi)\in\partial\sqcup\!\!\lrcorner^*_{C}(p-p_0,\alpha-\alpha_0), \\ \sigma\in\partial\Phi(e), \\ \alpha\in\partial\Pi^*(\chi), \\ u\in\partial\Gamma^*(f), \end{cases}$$

in terms of the values of the fields at the end of the finite step.

Introducing the dual product spaces $W = \mathscr{U}\times\mathscr{S}\times\mathscr{D}\times X\times\mathscr{D}\times X'\times\mathscr{F}$ and $W' = \mathscr{F}\times\mathscr{D}\times\mathscr{S}\times X'\times\mathscr{S}\times X\times\mathscr{U}$, we can arrange this set of relations to build up a global multi-valued *structural operator* $\mathbf{S}: W\mapsto W'$ governing the whole problem:

$$0\in\mathbf{S}(w) = \hat{\mathbf{S}}(w)+a, \quad w\in W, a\in W',$$

whose explicit form is given by:

$$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix} \in \begin{bmatrix} 0 & T' & 0 & 0 & 0 & 0 & -I_{\mathscr{F}} \\ T & 0 & -I_{\mathscr{D}} & 0 & -I_{\mathscr{D}} & 0 & 0 \\ 0 & -I_{\mathscr{S}} & \multicolumn{2}{c}{\partial\sqcup\!\!\lrcorner^*_{C}} & 0 & 0 & 0 \\ 0 & 0 & & & 0 & -I_{X'} & 0 \\ 0 & -I_{\mathscr{S}} & 0 & 0 & \partial\Phi & 0 & 0 \\ 0 & 0 & 0 & -I_X & 0 & \partial\Pi^* & 0 \\ -I_{\mathscr{U}} & 0 & 0 & 0 & 0 & 0 & \partial\Gamma^* \end{bmatrix} \begin{bmatrix} u \\ \sigma \\ p-p_0 \\ \alpha-\alpha_0 \\ e \\ \chi \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ -p_0 \\ 0 \\ 0 \\ 0 \\ -\alpha_0 \\ 0 \end{bmatrix}.$$

The conservativity of the operator $\hat{\mathbf{S}}$ follows from the duality existing between the pairs (T,T'), $(I_{\mathscr{U}},I_{\mathscr{F}})$, $(I_{\mathscr{S}},I_{\mathscr{D}})$, $(I_X,I_{X'})$ and the apparent conservativity of the other relations. In fact the operator $\mathbf{S}$ can be expressed as the sum of Cartesian products of monotone conservative operators and hence (see Section 2.2) turns out to be conservative.

The associated potential can then be evaluated by summing up the potentials of each component operator. Hence we can write:

$$\Omega(w) = \int_0^1 \langle\mathbf{S}(tw),w\rangle\,\mathrm{d}t = \int_0^1 \langle\hat{\mathbf{S}}(tw),w\rangle\,\mathrm{d}t - \langle\sigma,p_0\rangle - \langle\chi,\alpha_0\rangle,$$

to get:

$$\Omega(u,\sigma,p,\alpha,e,\chi,f) = \Phi(e)+\Pi^*(\chi)+\Gamma^*(f)+\sqcup_C^*(p-p_0,\alpha-\alpha_0)+\langle\sigma,Tu\rangle \\ -\langle f,u\rangle-\langle\sigma,e+p\rangle-\langle\chi,\alpha\rangle.$$

The functional Ω turns out to be linear in (u,σ), jointly convex with respect to the variables (p,α,e) and jointly concave with respect to the pair (χ,f).

According to the theory outlined in Section 2, the generalized gradient of Ω yields back the structural operator $\mathbf{S}$, i.e.

$$\mathbf{S} = \partial\Omega.$$

To achieve further evidence of this result, let us explicitly recover the operator form of the structural problem from the stationarity condition of Ω enforced at the point $w = (u,\sigma,p,\alpha,e,\chi,f)$:

$$0 \in \partial\Omega(w).$$

By virtue of the properties of Ω previously recalled, its stationarity is expressed by the following sets of relations:

$$\begin{cases} 0 \in \partial_u\Omega(w), \\ 0 \in \partial_\sigma\Omega(w), \\ (0,0,0) \in \partial_{(p,\alpha,e)}\Omega(w) = \partial_{(p,\alpha)}\Omega(w)\times\partial_e\Omega(w), \\ (0,0) \in \partial_{(\chi,f)}\Omega(w) = \partial_\chi\Omega(w)\times\partial_f\Omega(w). \end{cases}$$

Hence, performing the subdifferentials and the superdifferentials in the corresponding spaces, the operator form of the problem is recovered:

$$\begin{aligned} 0 \in \partial_u\Omega(w) &\Leftrightarrow f = T'\sigma, \\ 0 \in \partial_\sigma\Omega(w) &\Leftrightarrow e+p = Tu, \\ (0,0) \in \partial_{(p,\alpha)}\Omega(w) &\Leftrightarrow (\sigma,\chi) \in \partial\sqcup_C^*(p-p_0,\alpha-a_0), \\ 0 \in \partial_e\Omega(w) &\Leftrightarrow \sigma \in \partial\Phi(e), \\ 0 \in \partial_\chi\Omega(w) &\Leftrightarrow \alpha \in \partial\Pi^*(\chi), \\ 0 \in \partial_f\Omega(w) &\Leftrightarrow u \in \partial\Gamma^*(f). \end{aligned}$$

Reverting the steps above, a solution of the finite-step elasto-plastic problem can be shown to make the functional Ω stationary.

We then infer the following:

Proposition 1. A vector w is a stationarity point for the functional Ω if and only if it is a solution of the finite-step elasto-plastic structural problem.

6. MINIMUM PRINCIPLES

A family of functionals can be derived from the potential Ω by enforcing the fulfilment of field equations and of constitutive relations. All these functionals do assume the same value when evaluated in correspondence to a solution of the structural elasto-plastic problem.

We shall explicitly derive here a functional jointly convex in (u,p,α) which provides a useful tool for computational algorithms.

To this end we first recall that, by Fenchel's equality, the following equivalences hold (Rockafellar, 1970):

$$u \in \partial\Gamma^*(f) \Leftrightarrow \Gamma(u) + \Gamma^*(f) = \langle f, u \rangle,$$

$$\alpha \in \partial\Pi^*(\chi) \Leftrightarrow \Pi(\alpha) + \Pi^*(\chi) = \langle \chi, \alpha \rangle.$$

Substituting these relations in the expression of the functional Ω and enforcing in addition the kinematic compatibility condition $e = Tu - p$, we get:

$$\mathbf{\Omega}_1(u,p,\alpha) = \Phi(Tu-p) - \Gamma(u) - \Pi(\alpha) + \sqcup_C^*(p-p_0, \alpha-\alpha_0),$$

which is jointly convex in (u,p,α).

We can then state the following:

Proposition 2. A triplet (u,p,α) is an absolute minimum point for the convex functional $\mathbf{\Omega}_1$ if and only if it is a solution of the finite-step elasto-plastic structural problem.

In order to provide an explicit proof of this proposition, we will show hereafter that the functional $\mathbf{\Omega}_1$ turns out to be the potential of an operator associated with an alternative formulation of the finite-step elasto-plastic structural problem.

Let us consider the subdifferential of the convex functional $\mathbf{\Omega}_1$:

$$\mathbf{S}_1(u,p,\alpha) = \partial\mathbf{\Omega}_1(u,p,\alpha).$$

In explicit form the multi-valued operator $\mathbf{S}_1$ is given by:

$$\mathbf{S}_1(u,p,\alpha) = \begin{bmatrix} \\ \mathbf{A}(u,p) \\ \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \mathbf{B}(p,\alpha) \\ \end{bmatrix} - \begin{bmatrix} \partial\Gamma(u) \\ 0 \\ \partial\Pi(\alpha) \end{bmatrix},$$

where

$$\mathbf{A}(u,p) = \partial(\Phi \circ N)(u,p), \quad \mathbf{B}(p,\alpha) = \partial \sqcup_C^*(p-p_0, \alpha-\alpha_0),$$

and the operator $N: \mathscr{U} \times \mathscr{D} \mapsto \mathscr{D}$ is defined in such a way that $(\Phi \circ N)(u,p) = \Phi(Tu-p)$, that is:

$$N = [T, -I_{\mathscr{D}}].$$

The multi-valued operator $\mathbf{S}_1$ is apparently conservative and the integral theorem provided in Romano *et al.* (1994) ensures that $\mathbf{\Omega}_1$ is the potential of $\mathbf{S}_1$.

Hence any minimum point (u,p,α) of $\mathbf{\Omega}_1$ is characterized by the inclusion:

$$(0,0,0) \in \mathbf{S}_1(u,p,\alpha),$$

and vice versa.

To prove the result we have just to show that the operator $\mathbf{S}_1$ corresponds to a suitably reduced formulation of the finite-step structural problem expressed in terms of the state variables (u,p,α).

Actually the condition $(0,0,0) \in \mathbf{S}_1(u,p,\alpha)$ is equivalently expressed by the set of relations:

$$0 \in T' \, \partial\Phi(Tu-p) - \partial\Gamma(u),$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \partial \sqcup\!\!\lrcorner_C^*(p-p_0, \alpha-\alpha_0) - \begin{bmatrix} \partial\Phi(Tu-p) \\ \partial\Pi(\alpha) \end{bmatrix},$$

since

$$\mathrm{A}(u,p) = \partial(\Phi \circ N)(u,p) = N' \, \partial\Phi(N(u,p)) = \begin{bmatrix} T' \, \partial\Phi(Tu-p) \\ -\partial\Phi(Tu-p) \end{bmatrix},$$

and the operator $N' : \mathscr{S} \mapsto \mathscr{F} \times \mathscr{S}$, dual of N, is given by:

$$N' = \begin{bmatrix} T' \\ -I_{\mathscr{S}} \end{bmatrix}.$$

We may then conclude that there exists an external force satisfying the external constraint condition:

$$f \in \partial\Gamma(u),$$

which is equilibrated by an internal stress:

$$\sigma \in \partial\Phi(e),$$

such that the elastic strain e is associated with the pair (u,p) by the relations:

$$\varepsilon = Tu, \quad e = \varepsilon - p.$$

Moreover there exists a pair (σ, χ) satisfying the condition:

$$\begin{cases} \sigma \in \partial\Phi(e), \\ \chi \in \partial\Pi(\alpha), \end{cases}$$

such that:

$$(\sigma, \chi) \in \partial \sqcup\!\!\lrcorner_C^*(p-p_0, \alpha-\alpha_0).$$

The whole elasto-plastic finite-step problem is thus recovered. The converse implication follows at once by reverting the steps above.

Remark

Minimum principles in structural mechanics are especially relevant in twofold directions; solution techniques based on minimization procedures can be exploited and existence and uniqueness results can be provided under suitable conditions by making recourse to the mathematical methods of functional analysis.

In particular uniqueness is guaranteed if the functional to be minimized turns out to be strictly convex. In finite-step elastoplasticity the question of existence is by far more involved and, in general, still an open problem.

A discussion on this issue is definitively out of the scope of this paper. Anyway we may quote two recent contributions by Reddy and Griffin (1988), Reddy (1991) concerning the existence results for elasto-plastic models of the type proposed by Martin (1981) and Martin and Reddy (1988). Martin's model can be shown to be recovered as a special case of the *generalized standard* material considered here by assuming a cylindrical elastic domain in the product space of stresses and thermodynamic forces [see Romano *et al.* (1993b)].

6.1. *Sublinear yield modes*

In view of a direct comparison with an analogous minimum principle presented in literature (Comi and Maier, 1992; Comi *et al.*, 1992) we shall further specialize the expression of the dissipation functional $\sqcup\!\!\lrcorner_C^*$ to the case of sublinear yield modes.

Let the elastic domain be defined, for all $x \in V$, by means of a finite family of sublinear yield modes $\psi_i : \mathcal{S}_l \times X'_l \mapsto \Re \cup \{+\infty\}$ and yield limits Y_i:

$$C_l = \{[\sigma(x), \chi(x)] \in \mathcal{S}_l \times X'_l \mid \psi_i[\sigma(x), \chi(x)] \leqslant Y_i, \quad i = 1, \ldots, n\}.$$

The corresponding global sublinear yield modes Ψ_i are then defined by:

$$\Psi_i(\sigma, \chi) = \sup_{x \in V} \{\psi_i[\sigma(x), \chi(x)]\},$$

so that the global elastic domain is given by:

$$C = \bigcap_{i=1}^{n} C_i \quad \text{where} \quad C_i = \{(\sigma, \chi) \in \mathcal{S} \times X' \mid \Psi_i(\sigma, \chi) \leqslant Y_i\}.$$

Collecting the yield limits Y_i and the parameters λ_i in the vectors Y and λ, the expression of the global dissipation functional $\sqcup\!\!\lrcorner_C^*$ can be directly obtained by means of a formula provided in the appendix:

$$\sqcup\!\!\lrcorner_C^*(p-p_0, \alpha-\alpha_0) = \inf_{\lambda} \left\{ \lambda * Y \mid \lambda \geqslant 0, \begin{bmatrix} p-p_0 \\ \alpha-\alpha_0 \end{bmatrix} \in \sum_{i=1}^{n} \lambda_i \, \partial\Psi_i(0,0) \right\},$$

where

$$\lambda * Y \overset{\text{def}}{=} \sum_{i=1}^{n} \lambda_i Y_i, \quad \text{and} \quad \partial\Psi_i(0,0) = Y_i C_i^0.$$

The convex sets C_i^0 are the polars of C_i:

$$C_i^0 \overset{\text{def}}{=} \{(p, \alpha) \in \mathcal{D} \times X \mid \langle \sigma, p \rangle + \langle \chi, \alpha \rangle \leqslant 1 \quad \forall (\sigma, \chi) \in C_i\}.$$

Proposition 2 can then be re-formulated as follows:

Proposition 3. A quartet (u, p, α, λ) is a solution of the convex optimization problem:

$$\min \left\{ \Omega_{1,1}(u, p, \alpha, \lambda) \mid \lambda \geqslant 0, \begin{bmatrix} p-p_0 \\ \alpha-\alpha_0 \end{bmatrix} \in \sum_{i=1}^{n} \lambda_i Y_i C_i^0 \right\},$$

with

$$\Omega_{1,1}(u, p, \alpha, \lambda) = \Phi(Tu-p) - \Gamma(u) - \Pi(\alpha) + \lambda * Y,$$

if and only if it is a solution of the finite-step elasto-plastic structural problem.

Remark

In recent papers, G. Maier, C. Comi and U. Perego have investigated the formulation of variational principles for finite-step solutions of elasto-plastic structural problems (Comi and Maier, 1992; Comi *et al.*, 1992) assuming a constitutive model which is a special case of the one considered here. The time integration of the constitutive law has been performed according to a fully implicit scheme.

A minimum principle in (u, p, α, λ) is proposed in Comi *et al.* (1992) to characterize a solution of the problem; increments of the state variables in each finite step, instead of their final values, are taken as unknowns and a linear elastic behaviour is considered.

In particular the assumptions of sublinearity of the yield modes Ψ_i and their differentiability (even twice in the sufficiency proof) appear essential to their arguments.

In our notations the minimum principle proposed in Comi *et al.* (1992) reads:

$$\min \{\Omega_M(u, p, \alpha, \lambda) = \tfrac{1}{2}\|e\|_E^2 - \langle l, u\rangle - \Pi(\alpha) + \lambda * Y\},$$

subject to the constraints:

$$\lambda \geqslant 0, \quad \begin{bmatrix} p - p_0 \\ \alpha - \alpha_0 \end{bmatrix} = \sum_{i=1}^{n} \lambda_i \begin{bmatrix} \mathrm{d}_\sigma \Psi_i(\sigma, \chi) \\ \mathrm{d}_\chi \Psi_i(\sigma, \chi) \end{bmatrix},$$

where

$$Tu = e + p, \quad \sigma = Ee \quad \text{and} \quad \chi = \mathrm{d}\Pi(\alpha).$$

With reference to this minimum principle some observations are in order.

The functional $\Omega_{1,1}$ of our Proposition 3 coincides with Ω_M under the assumption of linear elastic behaviour, external frictionless bilateral constraints and differentiability of the functional Π, i.e.

$$\Phi(Tu - p) = \tfrac{1}{2}\|Tu - p\|_E^2, \quad \Gamma(u) = \langle l, u\rangle \quad \text{with} \quad u \in L_0.$$

There is however a significant difference between the two minimum principles for what concerns the second constraint condition.

First we observe that this condition on the functional Ω_M must be suitably re-written since the assumed differentiability of the yield modes Ψ_i is unnecessary and even questionable. In fact sublinear functionals turn out to be inherently non-differentiable at the origin. The notion of differentiability has thus to be replaced by the weaker notion of sub-differentiability.

Accordingly, the constraint condition proposed in Comi *et al.* (1992), must be re-written as follows:

$$\begin{bmatrix} p - p_0 \\ \alpha - \alpha_0 \end{bmatrix} \in \sum_{i=1}^{n} \lambda_i \, \partial\Psi_i(\sigma, \chi) \quad \text{with} \quad \sigma = Ee; \quad \chi = \mathrm{d}\Pi(\alpha).$$

Anyway, a comparison with the result of Proposition 3, reveals that the requirement above is unduly restrictive; in fact it can be recovered as a natural property of a solution of the extremum principle, formulated under a less stringent condition. In fact sublinear functionals enjoy the following characteristic property:

$$\partial\Psi_i(\sigma, \chi) \subseteq \partial\Psi_i(0, 0) \quad \forall\, (\sigma, \chi) \in \mathscr{S} \times X'.$$

A subtle point is worth being further clarified. As shown above the feasible set of the extremum principle proposed in Comi *et al.* (1992) turns out to be a proper subset of the variationally consistent one derived in the present paper.

This proper subset contains, however, the whole solution set since the conditions supplementing the functional Ω_M must be fulfilled by any solution of the problem. As a consequence, the principle claimed in Comi *et al.* (1992) does in fact hold true but its statement, being *non-optimal*, belongs to the category of *variational crimes*.

Finally it has to be observed that, due to the involved form of their constraint condition, the minimum principle proposed in Comi *et al.* (1992) has been referred to by the authors as *non-convex*. In this respect, we feel that a global minimum property stated in a non-convex context should, *legitima suspicione*, be carefully checked.

7. CONCLUSIONS

The variational flaws pointed out above suggest some general considerations.

Variational principles that are built up without a well established directory plan can eventually give rise to *ill-constraining* phenomena since constraint conditions in surplus to the minimal set are added. In this case poorer results are obtained and unduly complicated algorithms will be generated.

With specific reference to the present context the source of troubles can be clearly detected ; it is imputable to the mathematical theory resorted to. Classical smooth potential theory does not in fact provide adequate tools to handle with inherently non-smooth problems.

Up to now, travellers without non-smooth potential tools in their luggage, were compelled to discover variational principles for non-smooth problems by skilful intuition and to infer their validity *a posteriori* on the basis of *ad hoc* procedures. Often, on establishing the *if* and *only if* parts of the statement, different paths had to be followed and different assumptions had to be made to reach the goals ; sometimes differential rules were applied, despite their apparent inapplicability, as a last resource.

As a result the *pilot* could eventually lose the control of the navigation allowing for *clandestine* conditions to be shipped in.

The brand-new non-smooth potential theory provides now general guidelines for a direct variational formulation of problems initially put forth in terms of multi-valued operators.

Plastic flow problems are significant examples of this kind in structural mechanics.

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APPENDIX

We present here an explicit proof of the formula:

$$\sqcup_C^*(p-p_0,\alpha-\alpha_0) = \inf_\lambda \left\{\lambda * Y \mid \lambda \geqslant 0, \begin{bmatrix} p-p_0 \\ \alpha-\alpha_0 \end{bmatrix} \in \sum_{i=1}^n \lambda_i \partial\Psi_i(0,0) = \sum_{i=1}^n \lambda_i Y_i C_i^0 \right\}.$$

To this end we recall that the Minkowski formula associates a l.s.c. non-negative sublinear functional $\gamma_K : \mathscr{X} \mapsto \Re \cup \{+\infty\}$ with any closed convex set K containing the origin (Rockafellar, 1970):

$$\gamma_K(z) = \inf\{\mu \geqslant 0 : z \in \mu K\}.$$

Moreover, denoting by K^0 the polar of K, i.e. the closed convex set defined by:

$$K^0 \overset{\text{def}}{=} \{z^* \in \mathscr{X}' \mid \langle z^*, z\rangle \leqslant 1, \quad \forall z \in K\},$$

it results (Rockafellar, 1970):

$$\sqcup_K^*(z^*) = \gamma_{K^0}(z^*).$$

Further, there is a one-to-one correspondence between sublinear functionals $g_i : \mathscr{X} \mapsto \Re \cup \{+\infty\}$ and closed convex set $A_i \subseteq \mathscr{X}'$ according to the following formulae:

$$g_i(z) = \sup\{\langle z^*, z\rangle \mid z^* \in A_i\},$$

and

$$A_i \overset{\text{def}}{=} \{z^* \in \mathscr{X}' \mid g_i(z) \geqslant \langle z^*, z\rangle \quad \forall z \in \mathscr{X}\} = \partial g_i(0).$$

The next lemma shows that, when a convex set K is assigned as intersection of level sets of sublinear functionals g_i, its polar K^0 can be expressed in terms of the convex sets A_i corresponding to g_i.

Lemma. Let $g_i : \mathscr{X} \mapsto \Re \cup \{+\infty\}$. $i \in I = \{1, \ldots, n\}$ *be a family of non-negative sublinear functionals and K the convex set defined by:*

$$K = \bigcap_{i=1}^n K_i,$$

where

$$K_i = \{z \in \mathscr{X} \mid g_i(z) \leqslant a_i\},$$

are the level sets of g_i *at* $a_i > 0$. *Then*

$$K^0 = \left\{\text{conv}\left[\frac{A_i}{a_i}, i \in I\right]\right\} \overset{\text{def}}{=} \left\{\sum_{i=1}^n \xi_i \frac{A_i}{a_i} \mid \xi_i \geqslant 0 (i \in I), \sum_{i=1}^n \xi_i = 1\right\}.$$

Proof. Taking account of the expression of g_i in terms of A_i, each convex set K_i can be expressed in the form:

$$K_i = \{z \in \mathscr{X} : \langle z^*, z\rangle \leqslant a_i \quad \forall z^* \in A_i\}.$$

By definition of the polar set, it results:

$$K_i = a_i A_i^0 = \left[\frac{A_i}{a_i}\right]^0 \quad \text{and} \quad A_i = a_i K_i^0,$$

the second relation holding since g_i is non-negative and hence each set A_i is closed and contains the origin.

Let us recall the general relation:

$$\bigcap \left\{\left[\frac{A_i}{a_i}\right]^0, i \in I\right\} = \left\{\operatorname{conv}\left[\frac{A_i}{a_i}, i \in I\right]\right\}^0.$$

The convex hull on the right-hand side is closed and contains the origin so that, taking the polar of both sides we get the result (Rockafellar, 1970).